THE METHODOLOGY OF INDIAN MATHEMATICS AND ITS CONTEMPORARY RELEVANCE

Abstract :

While considerable attention has been paid to the achievements of Indian mathematicians, by cataloguing the results and processes discovered by them, very little work seems to have been done on the methodological foundations of Indian mathematics, its under- standing of the nature of mathematical objects, the nature of mathematical knowledge and the methods of validation of mathematical results and processes. Traditionally such issues have been dealt with mainly in the so called 'commentarial literature' which (though it formed an important component in traditional learning) has been completely ignored by modern scholarship. An attempt is made in this paper to present a preliminary view of the methodology of Indian mathematics as may be gathered from the major commentarial works, the commentary Buddhivilasini of Ganesha Daivajna (c 1545) on Bhaskaracharya's Lilavati (c 1150) and the commentary Bijanavankura of Krishna Daivajna (c 1601) on Bhaskaracharya's Bijaganita (c 1150).

While these commentaries provide a general exposition of many basic aspects of Indian mathematics, their main emphasis is in presenting what they refer to as upapatti (roughly translatable as demonstration, proof) for every result and process enunciated in the original texts of Bhaskaracharya. We present a few examples of such upapattis including a long demonstration (involving a series of demonstrations of many intermediate results) of the so called kuttaka process for the solution of first order indeterminate equations. As we see, one main feature of the Indian upapattis as contrasted with the Greek or the modern Western notion of 'proof is that the former does not involve any reference to formal deductions performable from some fixed set of axioms. This appears to be closely linked with the fact that in the Indian tradition mathematical knowledge is not viewed to be in any fundamental sense distinct from that in natural sciences. The Indian mathematicians declare that the purpose of upapatti is to clarify, disambiguate, remove all confusions etc, and to convince the fellow mathematicians of the validity of a result.

The Indian epistemological view point appears to be radically different from the standard Greek or modern Western view which seeks to establish mathematical knowledge as infallible absolute truth. Further, the Indian views concerning the nature of mathematical objects such as numbers etc., appear to be based on the framework developed by the Indian logicians and differs significantly at the foundational level from the set-theoretic universe of contemporary mathematics. It is argued that the Indian epistemological view point and the Indian views on the nature of mathematical objects etc, could contribute in a significant way to the development of mathematics today as they appear to have the potential of leading to an entirely new edifice for mathematics. A comprehension of the Indian methodology of mathematics would also help in making contemporary Indian mathematics come on its own and make its mark in the world of science.

I

Quite a few books have been written on the history of Indian tradition in mathematics (1). In addition, several general works on history of mathematics devote a section, sometimes even a chapter, to the discussion of Indian mathematics. While it is true that many of the results and processes discovered by the Indian mathematicians have been catalogued, there has been a total lack of attention to the methodology and foundations of Indian mathematics. There is very little discussion of the arguments by which Indian mathematicians arrive at and justify their results and [processes. In the same way no attention is paid to the philosophical foundations of Indian mathematics, its understanding of the nature of mathematical objects, the nature of mathematical knowledge and the nature of validation of mathematical results and processes.

Many of the commonly available books on history of mathematics declare or imply that Indian mathematics, whatever be its 'achievements, does not have any sense of logical rigor. It is p often said that no rigorous proofs are advanced in Indian mathematics (2). While the modern scholarship seems to be almost unanimous in holding the view that Indian mathematics totally lacks any notion of proof, a study of even the source works available in print would reveal that a great deal of emphasis is laid on providing what our Sastrakaras refer to as upapatti (roughly translatable as demonstration, proof etc.) for every result and process. In fact, some of these upapattis were noted in the various European works on Indian mathematics up to the first half of the nineteenth century (3). It would indeed be interesting to find out how the currently popular" view, that Indian mathematics lacks the very notion of proof, has come about during the last 100-150 years.

One of the major reasons for our total lack of comprehension, not merely of the notion of proof, but also of the entire methodology of Indian mathematics, is the scant attention we have so far paid to the source works themselves. It is said (4) that there are over one lakh manuscripts on Jyotihsastra which includes apart from ganita shandha (mathematics and mathematical astronomy) samhita skandha (omens) and hora (asrrology) also. Only a very small fraction of these texts have been published so far. The recently published 'Source Book of Indian Astronomy' (5), lists about 285 works published in mathematics and mathematical astronomy, of which about 50 are works written prior to the 12th century A.D., about 75 are works written during 12-15 centuries and about 165 are works written during 16-19th centuries. A serious study of even these published works would give us a reasonable idea of the methodology of Indian mathematics and astronomy.

In this context it is very important to realize that a great deal of methodological discussion is usually contained only in the so called 'commentarial literature' and is just briefly touched upon (if at all) in the so called 'original works'. While most of modern scholarship has concentrated on translating and analyzing a given work, without paying much heed to its commentaries (except in so far as to settle some points of controversy in fixing the data of the text),, it appears that traditionally the commentaries seem to have played at least as great a role in the exposition of the subject as the original text itself. It is no wonder that great mathematicians and astronomers such as Bhaskaracharya, Bhaskaracharya II, Ganesa Daivajna, Parames vara, Nilakantha Somasutvan, Muniswara, Kamalakara etc, not only wrote major original treatises of their own, but also took great pains to write erudite commentaries on either their own works or on important works of earlier scholars. It is in such commentaries that one finds detailed upavattis (demonstrations) of the results and processes discussed in the original text, as also a discussion of the various methodological and philosophical issues concerning Indian astronomy and mathematics. In the Appendix I we give a list of important commentorial works in Indian mathematics and astronomy, which are available in print. 'Even a cursory glance at these texts will show that the Indian mathematicians and astronomers have paid a great deal of atten Hon to the methodological foundations of their science and, more importantly, they lay great stress on providing elegant upapattis for every one of the results and processes discovered bythem.

It is true that none of the above published works has so far been translated into any of the Indian languages, or into English (6); nor have they been studied in any detail with a view to analyse the nature of mathematical arguments employed in the upapattis or to comprehend the methodological and philosophical foundations of Indian mathematics and astronomy. In this article we shall attempt to present Some examples of the kinds of upapattis provided in Indian mathematics, with particular reference to the commentaries of Ganesha Daivajna (c 1545) and Krishna Daivajna , (c 1601) on the texts Lilavati and Bijaganita respectively, of Bhaskaracharya II (c 1150). Apart from this, we shall also attempt some speculation on the philosophical foundations of Indian mathematics and its relation to other Indian sciences. More definitive understanding of the nature of Indian mathematics will call for a full-fledged study of all the source-works including a great mass of unpublished material. Such a study would not merely place our understanding of Indian mathematics in proper perspective, but also would help restore creativity to contemporary Indian mathematics by opening many new vistas especially at the foundational level.

II

According to Ganesha Daivajna, ganita the Indian science of mathematics is defined as follows: Ganyate sankkyayate tadganitam. Tatpratipadakatvena tatsamjnam sastram ucyate. Ganita is the (process) of calculation or numeration and the science which expounds this is also characterized by this name ganita. This ganita is mainly of two types: vyakla ganita and avyakta ganita.Vyaktaganita (also called pati ganita - calculation with the board), is that branch of ganita which employs clearly manifest procedures well known to the intelligent as well as the lay men for performing calculations. This is in contrast to avyakta ganita (also called bija ganita) which takes recourse to the use of avyakta or unknown (indeterminate, unmanifested, non-lapparent) quantities such as yavat tavat (as much as) kalaka (black) nilaka (blue) etc. These avyakta quantities are also called varna (colours) and are denoted by symbols ya, ka, ni, just as in modem algebra unknowns are denoted by symbols x,y, t etc.

The science of mathematics is generally taken to be a component part of Jyotihsastra (7). In fact, according to Nrisimhadaivajna (the Vartikakara for the autocomentary Vasanabhashya of Bhaskaracharya for Siddhantasiromani) the ganita skandha of Jyotihsastra is composed of four types of ganitas: vyakta ganita, avyakta ganita, graha ganita (mathematical astronomy which deals with calculation of planetary positions) and golaganita (spherical astronomy, which deals with demonstrations of calculation procedures using gola the sphere and vedha or observations). According to Ganesha Daivajna the prayojana (purpose) of the study of ganita sastra is 'the acquisition of knowledge concerning the orbits, raising, setting, measurements of sizes etc., of the planets, stars etc., and also the knowledge of samhita (omen) jataka (horOscopy) etc., which are the indicators of the merits and demerits earned via the deeds of former births'. The classification of ganita into avyakta and vyakta is dependent on whether or not indeterminate.quantities like yavat tavat etc., are employed in the various processes discussed. Thus vyakta ganita subsumes not only arithmetic and geometry, but also even topics included undc' 'algebra' such as solutions of equations, if one does not have to take recourse to introducing indeterminate quantities for carrying through the process of solution (8).

The most important feature of the commentaries such as those of Ganesha Daivajna and Krishna Daivajna is that they provide detailed demonstrations of every rule and procedure enunciated in the original text. The mode of exposition of these commentaries is as follows: In the beginning of each section, the context of the section is introduced along with its relation with what has already been expounded. Then each rule or process stated in the text is taken, its content explained in unambiguous terms and then a demonstration is provided justifying the given rule or procedure. The commentaries also deal with the appropriateness or otherwise of the way in which the exposition is organised in the original text, and any other important question that needs to be clarified in order to facilitate the understanding of the text. But they do not usually go into any discussion of further results they may be obtained as a consequence of the rules stated in the text. In fact the main purpose of a commentary such as the Buddhivilasini of Ganesha DaWajna on Lilavati is indeed one of providing upapattis.1 As Ganesha declares right at the beginning of his commentary:

"There is hardly any novelty in writing explanations for the extremely clear statements of Sri Bhaskara. Hence the knowledgeable mathematicians may take note of the specialty of my intellect in the statement of upapattis which is afterall the essence of the whole thing".

Amongst the works on Indian mathematics and astronomy which have been published, the earliest exposition of upapattis are those in astronomy by Bhaskarcharya (c 1150) found both in his commentary Vivarana on Sisyadhivriddhida-tantra of Lalla and his autocommen tary Vasanabhashya on his own Siddhantasiromani. Apart from these, Bhaskaracharya provides an idea of what is an upapatti in his own notes on Bijaganita in two places. In the chapter on Madhyamaharana (quadratic etc., equations) he poses the following problem:

"Say what is the hypotenuse of a plane figure, in which the side and upright are equal to fifteen and twenty? And show the upapattis (demonstration) of the received mode of computation".

Bhaskaracharya goes ahead and provides two upopattis which we shall discuss later) of this so called Pythagoras theorem. Again, towards the very end of the book in the chapter on Bhavita (equations involving products), while considering solutions (in integers) of equations of the form ax+by=cxy, Bhaskaracharya takes up an example and goes ahead to explain:

"The demonstration follows. It is twofold in each case: One geometrical and the other algebraic. The geometric demonstration is here delivered... The algebaric demonstration is next set forth..; This very operation has been delivered in a compendious form by ancient teachers. The algebraic demonstration must be exhibited to those who do not comprehend the geometric one. Mathematicians have declared algebra to be computation joined with demonstration: else there would be no difference between arithmetic and algebra. Therefore this explanation of the principle of resolution has been shown in two ways".

Clearly the tradition of exposition of upapattis is a much older one and Bhaskaracharya and the later mathematicians and astronomers are merely following the traditional practice of providing detailed upapattis in their commentaries and notes to earlier (or their own) works. These upapattis, as Bhaskaracharya has noted, are mainly of two types: Kshetragata upapatti or geometrical demonstration and avyakata ritya upapatti or algebraic demonstration (which could be rasigata or arithmetico- algebraic or varnagata or purely algebraic). To understand the nature of mathematical argumentation employed by Indian mathematicians, we shall in what follows present a few examples of upapattis as expounded by Ganesha Daivajna and Krishna Daivajna in their commentaries on Lilavati and Bijaganita of Bhaskaracharya. What we attempt is only a rought translation of the upapattis. These upapattis are in fact written in a technical Sanskrit (much like say the English of a modern text on Topology' in English) and hence these translations need to be improved upon after undertaking a systematic study of all these upapattis.

(1) The rule for calculating the square of a number:

According to Lilavati, the multiplication of two like numbers Hogether is the square. The square of the last digit is to be placed over it, and the rest of the digits doubled and multiplied by the last to be placed above them respectively;then repeating the number, except the last digit, again perform the like operation....' (see the example below):

Ganesha's upapatti for the above rule is as follows:

"By using the rule on multiplication, keeping in mind the place-values, and by using the mathematics of indeterminate, let us take a number with three digits with 'yd at the 100th place 'ka' at the 10th place and 'hi' at the unit place. The number is then (in the Indian notation with the 'plus' sign understood) ya 1 ka 1 ni 7. Using the rule (of Bijaganita) for the multiplication of indeterminate quanti ties, the square (of the above number) will beyaval yaka bha 2 ya ni bha 2kaval kani bha 2 ni va 1 (using the Indian notation, 'va' after a symbol standing for varga or square and 'bha' after two symbols standing for bhavita or product). Here we see in the ultimate place, the square of the first digit 'ya'; in second and third places there are 'ka' and 'ni' multiplied by twice the first 'ya'. Hence the first part of the rule "The square of the last digit..." In the fourth place we have square of 'hf. In the fifth we have 'ni' multiplied by twice 'ka'. In the sixth we have square of 'ni'. Thus we have derived the rest of the rule "Then repeating the number, excepting the last digit, again perform the last operation".

While Ganesha provides such avyaktaritya upapattis or algebraic demonstrations for all procedures employed in arithmetic, Sankara Variar in his commentary Kriyakramakari presents exclusively kshetragata upapattis or geometrical demonstrations for all such procedures.

(2) The theorem on the square of the diagonal of a right angled triangle: (generally referred to as the Pythagoras theorem):

Ganesha provides two upapattis for this which are the same as the ones outlined by Bhaskaracharya in his notes (vasana) on Bijaganita, and were referred to earlier. The first one is the upapatti involving the avyakata method and proceeds as follows:

Take the hypotenuse (karna) as the base and assume it to be 'ya' as in the figure. Let the bhuja and koti (the two sides) be 3, 4 respec tively. Let the perpendicular to the hypotenuse from the opposite vertex be drawn. This divides the triangle into two triangles which are similar to the original. Now the rule of proportion (anupata). When 'ya' is the hypotenuse the bhuja is 3, then when this bhuja 3 is the hypotenuse, the bhuja, which is now the segment of the hypotenuse to the side of the HoriginaU bhuja will be 3.3 ya. Again when 'ya' is the hypotenuse, the koti is 4, then when the koti 4 is the hypotenuse, the koti, which is now the segment of hypotenuse to the side of the (original) tod* will be 4.4 ya. Adding the two segments (abadhas) of ya the hypotenuse and equating the sum to (the hypotenuse) ya ... we get ya = 5'(9).

The other upapatti that Ganesha gives is the kshetragata or the geometrical upapatti which is as follows:

Take four triangles identical to the given and making different bhujas rest on different kotis form the square as shown. The interior square has for its side the difference of bhuja and koti. The area of each triangle is half the product of bhuja and koti and four times this added to the area of the total figure. This, by the rule... is nothing but the sum of the squares of bhuja and koti... The square root of that is the side of the (big) square which is nothing but the hypotenuse'.

The above example serves to indicate what Indian mathematicians mean by avyaktaritya upapatti and kshetragata upapatti.

(3) The rule of signs in algebra:

One of the crucial aspects of Indian mathematics is that in many upapattis the nature of the underlying mathematical objects plays a crucial role. We can for instance, refer to the upapatti given by Krishna Daivajna for the wellknown rule of signs in Algebra. While providing an upapatti for the rule, "the number to be substracted if positive (dhana) is made negative (rim) and if negative is made positive", Krishna Daivajna states:

'Negativity (rinatva) here is of three types - spatial, temporal and object-wise. In each case, (negativity) is indeed the vaiparitya or the oppositeness.... For instance, the second direction lying along the same line is called the opposite direction (viparita dik); just as West is the opposite of East.... Further, between two stations if one way of traversing' is considered positive then the other would be considered negative... In the same way, when one possesses said objects they would be called his dhana (or wealth). The opposite would be the case when another owns these objects... With this understanding of positivity (dhanatva) and negativity (rinatva) we now proceed to state the upapatti of the above rule as follows...'

Krishna Daivajna goes on to explain how the distance between a pair of stations can be computed knowing that between each of these statios and some other station on the same line. Using this he demonstrates the above rule that "the number to be substracted if positive is made negative...." (10).

(4) The Kuttaka process for the solution of linear indeterminate equations:

To really understand the nature of upapatti in Indian mathematics one will have to analyse some of the real lengthy demonstrations given by them for the more complicated results and processes. One will also have to analyse the sequence in which the results and the demonstrations are arranged to understand their method of exposition and logical sequence of argumentation. One can quote many examples of such long and sequential demonstrations in the works cited above. We shall consider the systematic derivation given by Krishna Daivajna for the entire Indian process of kuttaka employed to solve first order indeterminate equation of the form (ax+c)/b - y, where a, b, c are given integers and x, y are to be solved for in integers (11).

Since this upapatti is rather lengthy, it is presented separately as Appendix n. We may here merely recount the essential steps. Krishna Daivajna first shows that the solutions for x,y do not vary if we factor all the three numbers a, b and c, by the same (common) factor. He then* shows that if 'a' anil V have a common factor then the above equation will not have a solution unless 'c' is also divisable by the same common factor. Then follows the upapatti of the process of finding the greatest common factor of 'a' and 'b' by mutual division (the so called Euclidean algorithm). He tnen provides an upapatti for the kuttaka method of finding the solution by using the quotients obtained in the above mutual division, based on a detailed analysis of the various operations in reverse (vyasta vidhi).

Finally he shows why the process differs slightly depending upon whether there are odd or even number of coefficients generated in the above mutual division!

III

What seems to be all too apparent from any study of the upapattis in Indian mathematics is that the notion of upapatti is significantly different from the notion of proof as understood in the Greek and the modern Western traditions in mathematics. This has been for instance noted by Saraswati Amma, who remarks that (12):

"There was an important difference between the Indian proofs and their Greek counterparts. The Indian's aim was not to build up an edifice of geometry on a few self-evident axioms, but to convince the intelligent student of the validity of the theorem so that visual demonstration was quite an accepted form of proof... another characteristic of Indian mathematics which makes it differ profoundly from Greek mathematics. Knowledge for its own sake did not appeal to the Indian mind. Every discipline (Sastra) must have a purpose".

In what follows we shall elaborate on the above very perceptive remarks of Saraswati Amma.

The upapattis of Indian mathematics are presented in a technical or precise language and very carefully display all thestepsof the argument as also alt the general principles which are employed. In this sense they are no different from the 'proofs' pund in Greek or modern Western mathematics. But what is peculiar to the upapattis of Indian mathematics is that while presenting the argument in an 'informal' manner (which is common in mathematical discourse anyway) they make no reference what-so- ever to any fixed set of axioms or link the given argument to 'formal deductions' performable from such axioms. The upapattis of Indian mathematics are not formulated with any reference to a formal deductive system.

Most of the mathematical discourse in the Greek as well as modern Western tradition is carried out with clear reference to some formal deductive system, though the discourse it sell might be in the 'informal' mode, a la Indian mathematics. More importantly the ideal view of mathematics in both the Greek and modern Western tradition is that of a formal deductive system. Their view is that 'real mathematics' is (and ought to be presented) as formal derivations from formally stated axioms. This ideal view if mathematics is intimately linked with yet another major philosophical presupposition of Western tradition - that mathematics constitutes a body of infallible or absolute truths. Perhaps it is only the ideal of a formal deductive system which could presumably measure up to this other ideal of mathematics being a body of absolute truths. It is this quest for securing absolute certainty to mathematical knowledge, which has motivated most of the foundational and philosophical investigations into mathematics and has also shaped the entire course of mathematics in the Western tradition, right from the Greeks to the contemporary times (13,14).

What the upapattis of Indian mathematics reveal is that the Indian epistemological position on the nature and validation of mathematical knowledge is very different from that in the Western tradition. This is brought out for instance by the understanding held amongst the Indian mathematicians as to what a upapatti is supposed to achieve. Ganesha Daivajna declares in his preface to the commentary on Bhaskaracharya's Lilavati that:

"Whatever be stated in the vyakta or avyakta branches of mathematics without upapatti it will not be ren dered nirbhranta (free from misapprehension). It will not acquire any standing fin an assembly of scholarly mathematicians.' The upapatti is directly perceivable like a mirror in hand. It is therefore, as also for the elevation of the intellect (buddhi vriddhi), that 1 proceed to enunciate upapatti in its entirety".

As the above statement shows, the purpose of upapatti seems to have been (i) to remove misapprehension and confusion in the interpretation and understanding of the result or process expounded in the text; (ii) to satisfy the scholarly community of mathematicians (that the result is acceptable and the person who enunciates it is competent) and (iii) to enhance the powers of intellect. Clearly the purposes for providing upapattis appear to be very different from what a 'proof in Western tradition of mathematics is supposed to do, namely to establish once and for all the 'absolute truth' of a given proposition.

In this context it is perhaps important to emphasize that in the Indian tradition, mathematical knowledge is not taken to be different in any 'fundamental sense' from that in all other natural sciences. The methods for acquiring valid knowledge in mathematics are the same as in all other sciences, namely via the Pramanas: Pratyaksha (perception) Anumana (inference) Sabdaor Agama (tradition or textual) etc. In fact Ganesha's statement regarding the role of upapattis in vyakta and avyakta ganita is a rephrasing of an earlier statement of Bhaskaracharya on the role of upapattis in mathematical astronomy (grahaganita). In the beginning of the goladhvya of Siddhantasiromani, Bhaskaracharya states:

'Starting from the Madhyamaadhikara (the beginning of Grahaganitadhyaya of Siddhantasiromani) whatever be the ganita (calculational procedure) described here of the (motion of the) heavenly bodies, without its upapatti a mathematician will not acquire status in the scholarly assemblies; he will not himself be free of doubt (inissamsaya). Since this upapatti is easily perceivable in the (armiliary) sphere like a gooseberry in the hand, I therefore start golodhyaya (the section on Spherics) for the sake of expounding upapatti'.

'As the commentator Nrisimha Daivajna explains, 'the phala (utility) of upapatti is panditya (scholarship) and "also removal of doubts (for oneself) which would enable one to reject wrong interpretations made by others due to bhranti (misapprehension) or otherwise'.

In his auto commentary Vasanabhashya to Siddhantasiromani, Bhaskaracharya discusses what is the main source of valid knowledge (pramana) in mathematical astronomy. And he declares that 'Yadyevamucyate ganitaskandhe upapattimanagama eva pramanam' - 'Whatever is discussed in mathematical astronomy, the pramana is tradition or established text which can be supported by upapatti'. And of course here upapatti surely includes observation also, as it is clearly stated; say for instance, that the upapatti for the mean periods of planets involves observing it through instruments daily etc. Observation or experiment is also acceptable as upapatti in mathematics.

What emerge from all this is the following: The upapatti or demonstration in Indian mathematics serves to clarify and dis ambiguate the given result or process and to convince the lister nerjreader as to the validity of the result. In no sense is it iintended to be an approximation to some ideal of a mechanical or fool-proof way of establishing the absolute truth of the given result starting from a once for all given set of axioms. In fact the upapattis of Indian mathematics have a considerably flexible argumentational structure, depending on the context and purpose of enquiry, the result to be demonstrated, and the listener/ reader for whom the demonstration Is meant. While being contcx tual in this manner, the upapattis, are still highly technical and, whenever necessary,go into a great deal of hair-splitting argumentation. Further, they are written in a technical Sanskrit which, though not always as precise as the language of Navya- nyaya, is precise enough to carry on most of the mathematical argumentation as clearly as possible. It is no wonder then that Bhaskarcharya declares (in his Vasanabhashya) that upapatti can only be understood by one proficient in the Intricacies of language.

We have so far discussed how upapattis are not concerned with establishing the absolute truth of a result and hence are not formulated with reference to any formal deductive system. As regards the modes of argument which are allowed in the upapattis of Indian mathematics one distinctive feature appears to be that Indian mathematicians allow the method of indirect proof (reduction ad absurdum) only for showing the non-existence of certain entities, but not for proving the existence of an entity, which existence, is not demonstrable (at least in principle) by other (direct) means of proof. The method of indirect proof is called tarka by the Indian logicians. What the Indian mathematicians are following is only the general methodological dictum of most schools of Indian philosophy that tarka is not an independent pramana, and cannot be used to conclude the existence of an entity, which existence cannoi be otherwise proved (at least in principle) by the allowed pramanas.

We should here emphasise the fact that the Indian mathematicians do not eschew the method of indirect proof altogether. For instance let us consider the upapatti of the result that "a negative number has no square root". To show this Krishna Daivajna proceeds as follows:

'A negative number is not a square. Hence how can we consider its square root? It might however be argued that "why will a negative number not be a square? Surely it is not a royal fiat"... Agreed Let it be stated by you who claim that a negative number is a square as to whose square it is; surely not of a positive number, for the square of a positive number is always positive by the rule .... not also of a negative number. Because then also the square will be positive by the rule.... This being the case, we do not see by any means that number whose square becomes negative'.

In not accepting the method of indirect proof as a valid means for establishing existence of an entity (which existence is not even in principle establishable via direct means of proof), the Indian mathematicians tend to take what is now-a-days referred to as the constructivist approach to the issue of mathematical existence. But the Indian philosopher's logicians etc do much more than merely disallowing certain existence proofs. The general Indian philosophical position is in fact one of completely eliminating from logical discourse all reference to such aprasiddha (unlocatable) entities, whose existence is not even in principle accessible to direct means of verification. This appears to be also the position adopted by the Indian mathematicians. It is for this reason that many an "existence theorem" (where all that has been proved is that the non-existence of a hypothetical entity is incompatible with the accepted set of postulates) of Greek "or modern Western mathematics would not be considered significant or even meaningful by Indian mathematicians.

IV

The Indian epistemological view point on the nature of mathematical knowledge and its validation needs to be investigated in detail, as it could prove to be of great relevance for the development of mathematics today. Contemporary mathematics, being rooted entirely in the modem Western tradition, does suffer from serious limitations which can be traced to the kind of epistemology and philosophy of mathematics which have governed the development of mathematics in the Western tradition right from the Greek times. Firstly there is the perennial problem of "Foundations" posed by the ideal view of mathematical knowledge as a set of infallible of absolute truths, which is basic to the Western epistemology of mathematics. As is well known, the continued effort of mathematicians and philosophers of the West to secure for mathematics the status of indubitable knowledge has not succeeded; and there is perhaps a growing feeling that this goal may after all turn out to be impossible. However, within the Western philosophical tradition, it is not likely that any radically different epistemology of mathematics would get developed, and so the driving force for Western mathematics is likely to continue to be a search for absolute truths and modes of establishing them, in one form or the other. Surely this could lead to progress in mathematics, but it would be 'progress' of a limited kind and within the narrow confines of the Western quest for indubitable knowledge in the domain of mathematics (15).

Apart from the problems inherent in the very goals set for mathematics, there are also several other serious inadequacies in the Western epistemology and philosophy of mathematics which are now-a'-days being seriously discussed by many scholars. Most of these center around the issue that the ideal view of mathematics as a formal deductive system causes serious distortion in the very practice of the science of mathematics. Some scholars have argued that this ideal view of mathematics has rendered philosophy of mathematics totally barren and incapable of providing any understanding of the actual history of mathematics, the logic of mathematical discovery "and in fact the whole of creative mathematical activity (16). Consider for instance the contrast between the ideal view that a mathematical proof ought to be infallible and actual (and accepted) mathematical practice as portrayed in a recent book (17):

'On the one side, we have real mathematics, with proofs which are established by the "consensus of the qualified". A real proof is not checkable by a machine, or even by any mathematician not privy to the gestalt, the mode of thought of the particular field of mathematics in which the proof is located. Even to the "qualified reader" there are normally differences of opinion as to whether a real proof (i.e. one that is actually spoken or written down) is complete or correct. These doubts are resolved by communication and explanation, never by transcribing the proof into first order predicate calculus. Once a proof is "accepted", the results of the proof are regarded as true (with very high probability). It may take generations to detect an error in a proof... On the other side, to be distinguished from real mathematics, we have "met mathematics".... It portrays a structure of proofs which are .indeed infallible "in principle".... (The philosophers of mathematics seem to claim) that the problem of fallibility in real proofs.... has been conclusively settled by the presence of a notion of infallible proof in metamathematics... One wonders how they would justify such a claim".

Apart from the fact that the modern Western epistemology of mathematics fails to give any adequate account of the history of mathematics and current mathematical practice, there is also the growing awareness that the ideal of mathematics as a formal deductive system has had far reaching consequences in the teaching of mathematics. The formal deductive format adopted in most mathematics books and articles greatly hamper understanding and leave the student with no clear idea of what is being talked about (18). Still, if real mathematics is indeed formal derivations from formally stated axioms, then such understanding should indeed be sought elsewhere, and no reform need to be attempted in the style of mathematical discourse (19).

We wish to emphasize that the Indian epistemology of mathematics, if sufficiently researched upon by our present day scholars, may lead to a major revision of the current concepts on the nature of mathematical knowledge and its validation. Another important foundational issue in mathematics is that concerning the nature of mathematical objects. Here again the philosophical foundations of contemporary mathematics are extremely 'unsatisfactory' with none of the major schools of thought, namely Platonism, Formalism or Intuitionism, being able to give satisfactory account of what indeed is the nature of the objects (such as numbers) dealt with by mathematics and how they are related to (other) objects in the world (20).

For a discussion of the nature of mathematical objects as understood in the Indian tradition, we will have to see not only the texts in mathematics but also various texts of the different schools of Indian philosophy, wherein is debated the ontological status of mathematical objects such as numbers and their relation to other entities of the world. This would require a major investigation by itself. To give an idea of the isophistication of the Indian philosophy of mathematics in this regard, we will briefly outline the theory of numbers as developed for instance in the Nyaya-Vaiseshika school. As is well-known, in the Western tradition of mathematics, there was really no significant discussion on the nature of numbers after the Greek times, till the work of Frege in the nineteenth century. Recently there have been some investigations (21) which seem to indicate that the Nyaya theory of numbers might prove to be considerably superior in several respects to Frege's theory or the later developments which have followed from it (22).

In the Nyaya-Vaisesika ontology, samkhya or number-property is assigned to the category guna (roughly translated as quality) which resides in dravya (roughly translated as substance) via the relation samavaya - translated as inherence, which is also the relation between whole and parts, jati (genus or universal) and vyakti (species or individual) etc. This samavaya is the relation by which" a samkhya such as dvitva (two-ness or duality) is related to each of the objects of a pair, and gives raise to the jnand or cognition: 'Ayam dvitoavan' - This (one) is (a) locus of two-ness'. Apartjfrom this, the number-property, dvitva (two- ness) is related to both the objects together via a relation called 'paryapti (completion) and gives rise to the cognition 'Imau dvau' - 'These are two'. So according to Nyaya, there are two ways in which number-properties such as one-ness or unity, two-ness or duality, 'three-ness etc., are connected with things numbered- firstly via samavaya relation with each thing and secondly via paryapti'relation with the things together.

The paryapti relation connecting the number-property to the numbered things together is taken by the Naiyayikas to be a svarupa sambandha (or a self-linking relation), where the two terms of the relation is identified ontologically. Thus, according to the Naiyayikas any number property such us two-ness is not unique. There are indeed several two-nesses one associated (and identified) with every pair of objects (23). There are of course the universals such as dvitvatva (two-ness) which inhere in each particular two-ness associated (and identified) with each pair of objects.

The fact that the Naiyayikas talk of the relation paryapti by which number property such as two-ness resides in both the numbered objects together and not in each one of them, has led various scholars

to compare it with 'the Frege's theory of numbers. According to Bertrand Russel's version of this theory, there is a unique number two which is the set of all sets of two elements (or pair of objects). Thus the number two is a set of 'second-order' somewhat analogous to the universal two-ness (dvitvatva) of the Naiyayikas which may be thought of to be a property of 'second-order'.
The most crucial Way in which the Naiyayika theory differs from all the modern Western formulations is that the Naiyayikas talk only in terms of properties and that too with clearly specified ontological status, and totally avoid notions such as sets whose ontology is dubious (24). Any number property such as two-ness associated with a pair of objects is ontologically identified with the pair, or both the objects together, and not with any 'set' (let alone the set of all sets) constituted by such a pair.

The Naiyayika theory of number and paryapti is a highly sophisticated one and was developed during 16-I9th centuries in the context of some important issues in Indian logic. Its further development in the context of mathematics and its foundations are perhaps major challenges facing our contemporary scholars in logic, mathematics and philosophy. Apart from their theory of numbers, the general approach of the Indian logicians is what may be referred to as 'intentional' as opposed to the 'extensional' approach of most of Western logic and mathematics (25). It is precisely because of the fact that the Indian logicians have built a very powerful system of logic which is able to handle properties as they are (with both their intentions and extension) and not by reducing them to classes (which are pure extensions, with the intention being abstracted away), that there seems to be a great potential for the methodology of Indian logic in creating an entirely new edifice for mathematics. For, as is generally understood (26):

"Mathematics, as it exists today, is extensional rather than intentional. By this we mean that, when a prepositional function enters into a mathematical theory, it is usually the extension of the function (i.e. the totality of entities or sets of entities that satisfy it) rather than its intention (i.e. its "context" or meaning) that really matters. This leaning towards extensionality is reflected in a preference for the language of classes or sets over the formally equivalent language of predicates with a single argument...."

If the elementary propositions of the theory are of the form T(x)'(x has F' - where 'F' is predicate with single argument V which runs over a domain of 'individuals') then it is indeed true that it is but a matter of preference whether we use the language of predicates or of the classes (of all those individuals which satisfy the corresponding predicate). However the elementary propositions of Indian logic are of the form 'xRy' which relate any two 'entities' (not necessarily 'individuals') x, y via a relation R. The elementary proposition in Indian logic is always composed of a viseshya (qualificand x), viseshana or prakara (qualifier y) and a samsarga (relation R). Here y may also be considered as a dharma (property) residing in x via relation R. Using these and many other notions, the Indian logicians have developed a precise technical language, based on Sanskrit, which is unambiguous and makes transparent the logical structure of any (complex) proposition and which is Used in some sense like the symbolic formal languages of modern mathematical logic (27). The Indian logicians seem to have used this language mainly as a vehicle of conducting philosophical discourse concerning the nature of entities (padarthas) and their relations.

Now, as regards the nature of mathematics, the dominant view perhaps subscribed to by most contemporary mathematicians is essentially that adopted by Bourbaki, which may be stated as follows (28):

"... Mathematics is understood by Bourbaki as a study of structures, or systematic patterns of relations, each particular structure being characterized by a suitable set of axioms. In mathematics, as it exists at the present time, there are three great families of structures... namely algebraic structures, topological structures and ordinal structures. Any particular structure is to be thought of as inhering in a certain set E which functions as a domain of individuals for the corresponding theory".

It is for the above reason that, 'Bourbaki presents the whole of mathematics as an extension of the theory of sets' (29). Now if the study of abstract structures is indeed the goal of mathematics, there is no reason why this enterprise should necessarily be based on the theory of sets, unless one does not have the appropriate logical apparatus to handle philosophically more perspicuous notions such as properties, relations etc. As we stated earlier, the endeavor of the Indian logicians has been precisely one of developing such a logical apparatus. How powerful this apparatus is and how it could be employed to evolve a more comprehensive mathematics or theory of structures is very much a topic of speculation, if not serious investigation, in the years to come.

What we have indicated above are just a few examples (which are at this stage quite tentative and speculative) of how the methodology of Indian mathematics could turn out to be of considerable relevance for the development of mathematics today. For the Indian mathematicians' this could be of particular significance since the Indian tradition of mathematics, which was alive and fairly creative and dynamic till about two centuries ago, has yet to come on its own once again. It is all too apparent that while in the recent decades there has been a considerable revival of mathematical activity in India; it is yet to make any serious impact by way of significant contributions or new directions of research in mathematics. For Indian mathematics to come on its own, it would be necessary for us to rediscover our own genius in mathematics, as that is what would perhaps shape the nature and directions of our endeavors' in mathematics as it has done over the long history of Indian civilization. It is perhaps only this way that we can really comprehend the nature and potential of the achievements of even our twentieth century mathematicians, starting from the legendary Srinivasa Ramanujan (whose works have so far been analyzed solely in the context of and the from the stand point of the modem Western tradition in mathematics), and evolve a way of creating a dynamic tradition of mathematics once again in this country.

References and Footnotes

1. Some of the books are:

(a) B. B. Datta and A.N. Singh, History of Hindu Mathematics, Part I, II Lahore 1935,1938 (Rep. Delhi 1962)

(b) C.N. Srinivasa Iyengar, History of Indian Mathematics, Calcutta 1967

(c) A.K. Bag, Mathematics in Ancient and Medieval India, Varanasi 1979

(d) T.A. Saraswati Amma, Geometry in Ancient and Medieval India, Varanasi 1979.

2. The following quotation is adequate to bring out this commonly aired opinion:

'As our survey indicates, the Hindus were interested in and contributed to the arithmetical and computational activities of mathematics rather than to the deductive patterns. Their name for mathematics was ganita, which means "the science of calculation".

There is much good procedure and technical facility, but no evidence that they considered proof at all. They had rules, but apparently no logical scruples. Moreover, no general methods or new viewpoints were arrived at in any area of mathematics. It is fairly certain that the Hindus did not appreciate the significance of their own contributions. The few good ideas they had, such as separate symbols for the numbers, were introduced casually with no realization that they were valuable innovations. They were not sensitive to mathematical values. Along with the ideas they themselves advanced, they accepted and incorporated the crudest ideas of the Egyptians and Babylonians'.(Morris Kline: Mathematical Thought from Ancient to Modern Times, Oxford, 1972, p. 190).

3. For instance, Colebrooke's translation of portions of Brahmasphutasiddhanta of Brahmagupta and Lilavati and Bijaganita of Bhaskaracharya (published in London in 1817 under the title 'Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupta and Bhaskara') contains (as part of footnotes) sketches of many upapattis (provided by the commentators to these works) which are referred to as demonstrations. Another important notice of the fact that detailed proofs are provided in the Indian texts on mathematics, is due to Charles M. Whish, who in an important article in 1830 ('On the Hindu Quadrature of the Circle...' Trans. Roy. As. Soc. (G.B.) 3,1839, 509-523) pointed out that infinite series for π and for trigonometric functions were discussed in texts of Indian mathematics much before their 'discovery" in Europe. Whish concluded his paper with a sample proof from the Malayalam text Yuktibhasha (c 1608) of the theorem on the square of the diagonal of a right angled triangle (the so called Pythagoras theorem) and also promised that: 'A farther account of the

Yuktibhasha, the demonstrations of the rules for the quadrature of the circle by infinite series, with the series for the sines, cosines, and their demonstrations, will be given in a separate paper'. It appears however that Whish did not publish any further paper on this subject.
4. See for instance, D. Pingree, Jyotihsastra: Astral and Mathematical Literature, Wiesbaden 1981, p. 118.

5. K.V. Sarma and B.V. Subbarayappa, Indian Astronomy: A Source Book, Bombay 1985.

6. The book of Saraswati Amma (Ref Id) and the following works of C.T. Rajagopal and his collaborators provide an idea of the kind of upapattis that are presented in the Malayalam work Yuktibhasha of Jycstha Deva (c.1608) for various results in geometry, trigonometry and (hose concerning infinite series for the trigonometric functions and π :

(a) K. Mukunda Marar, Proof of Gregory's series, Teacher's Magazine 15,1940,28-34.

(b) K. Mukunda Marar and C.T. Rajagopal, On the Hindu Quadrature of the Circle, J.B.B.R.A.S. 20, 1944, 65- 82.

(c) C.T. Rajagopal, A Neglected Chapter of Hindu Mathematics, Scripta Mathematica 15, 1949, 201-209.

(d) A. Venkataraman, Some interesting proofs from Yuktibhasha, Math Student 16,1948,1-7.

(e) C.T. Rajagopal and A. Venkatarman, The Sine and Cosine Power Series in Hindu Mathematics, J.R.A.S.B. 15,1949,1-13.

(f) C.T. Rajagopal and T.V.V. Aiyer, On the Hindu Proof of Gregory's Series, Scripta Mathematical, 17,1951,65- 74.

(g) C.T. Rajagopal and M.S. Rangachari, On Medieval Kerala Mathematics, Archive for History of Exact Sc., 35(2), 986,91- 99.

7. It is not that the study of mathematics is completely tied up with astronomy alone. In fact the earliest texts dealing with geometry, the Sulva sutras (generally dated as being prior to 8th century B.C.) are part of Mpa (a vedahga different from jyotiska) and deal with the construction of altars. At a much later period the laina mathematician Mahavira (c 9th century) enumerates the uses of ganita as follows:

In all transactions which relates to wordly, vedic or other similar religious affairs calculation is of use. In the science of love, in the science of wealth, in music and in drama, in the art of cooking/in medicine, in architechture, in prosody, in poetics and poetry, in logic and grammar land such Other things, and in relation to all that t. constitutes the peculiar value of the arts, the science of calculation (ganita) is held in high esteem. In relation to the movement of the sun and other heavenly bodies, in connection with eclipses and conjunctions of planets, and in connection with the triprasna (direction, position and time) and the course of the moon-indeed in all these it is utilized. The number, the diameter and the perimeter of islands, oceans and mountains; the extensive dimensions of the rows of habitations and halls belonging to the inhabitants of the world of light, of the world of the gods and of the dwellers in hell, and other miscellaneous measurements of all sorts - all these are made out by the help of ganita. The configuration of living beings therein, the length of their lives, their eight attributes, and "other similar things; their progress and other such things, their staying together, etc. - all these are dependent upon ganita (for their due comprehension). What is the good of saying much? Whatever there is in all the three worlds, which are possessed of moving and non-moving beings, cannot exist as apart from ganita (measurement and calculation)'. (Passage from Ganitasarasathgraha of Mahaviracharya in cited B.B. Datta and A.N. Singh Ref (la), Vol. I p. 5).

8. The well-known text Lilavati of Bhaskaracharya II (c 1150) deals with vyakta ganita and is divided essentially into the following sections: (1) Paribhasha (units and measures) (2) Samkhyasthananirnaya (place value system) (3) Parikarama shtakani (eight operations of arithmetic, namely addition, subtraction, multiplication, division, square, square root, cube, cube root) (4)

Bhinnaparikarmashtakam (operations with fractions) (5) Sunyaparikarmani (operations with zero> (6) Prakirna (miscellaneous processes, including trairasika (rule of three) (7) Misravyavahara (investigation of mixture, ascertainment of composition as principal and interest joined and so forth) (8) Sredhivyavahara (progressions and series) (9) Kshetravyavahara (plane geometry) (10) Khatavyavahara (excavations and solids) (11) Citi, rakacha and rasi vyavahara (calculation with stacks, saw, mounds of grain) (12) Chayavyavahara (gnomonics) (13) Kuttake (linear indeterminate equations) (14) Ankapasa (combinatorics of digits).

The text Bijaganita of Bhaskaracharya deals with avyakata ganita and is essentially divided into the following sections: (1) Dhanarnashadvidham (the six operations with positive and negative quantities, namely addition, subtraction, multiplication, division, square and square root) (2) Khashadvidham (the six operations with zero) (3) Avyaktashadvidham (the six operations with indeterminate quantities) (4) Karanishadvidham (the six operations with surds) (5) Kuttaka (linear indeterminate equations) (6) Vargaprakriti (quadratic indeterminate equation of the form Nx2 + m = y2) (7) Chakravala (cyclic process for the solution of above quadratic indeterminate equation) (8) Ekavarnasamikarana Khanda (simple equations with one unknown) (9) Madhyamaharana (quadratic etc., equations) (10) Anekavarnasamikarana (simple equations with several unknowns) (11) Madhyamaharanasyabhedah (varieties of quadratics) (12) Bhavitam (equations involving products). Here the first seven sections, starting from dharnashadvidham to chakravala are said to be bijopayogi (adjuncts to analysis) and the last five sections deal with bija or analysis which is mainly of two types: Ekavarnasamikarana (equation with single unknown) and Anekavarnasamikarana (equation with several unknowns).

Ganesha Daivajna raises the issue as to the propriety of including discussion of kuttaka (linear indeterminate equations) and ankapasa (combinatorics) etc., in the work on vyaktagunita, Lilavati, as they ought to be part of Bijaganita. He then goes on to explain that this is alright as an exposition of these subjects can be given without employing avyaktamarga, i.e., procedures involving use of indeterminate quantities. An interesting discussion of the relation between vyakta and avyakta ganita is to be found in the commentary of Krishna Daivajna on Bijaganita of Bhaskaracharya. The statement of Bhaskara 'vyaktam avyakta-bijam' can be interpreted in two ways. Firstly that vyakta is the basis of avyakta (avyaktasya bijam) because till the knowledge of vyakta ganita (composed of addition, and other operations, the rule of three etc.) is not had, one cannot even think of entering into a study of avyaklaganita. It is also true that vyakta is that which is based on avyakta (avyaktam bijam yasya) because though the procedure of vyaktaganita do not depend upon avyakta methods for being carried through (svarupa nirvana), when it comes to justifying the vyakta methods by upapattis or demonstrations whole of vyakta ganita is dependent on avyakta ganita.

9. This method of proving the so called Pythagoras theorem using similar triangles seems to have appeared in Europe for the first time in the work of Wallis in the seventeenth century.

10. It would be interesting to compare this clear understanding of the rule of sighs in Indian mathematics with the kind of confusion that seemed to prevail on negative quantities in Europe even as late in the 18th century, as noted by Bourbaki (Theory of sets, Paris, 1968, p.314):

The embarrassment of algebraists in the presence of negative numbers vanished only when analytical geometry provided them with a convenient "interpretation". But, even in the eighteenth century d'Alembert (although a convinced "positivist"), when I discussing the question in the Encyclopedia, suddenly lost courage after a column of somewhat confused explanations and contented himself with concluding that the rules of algebraic operations on negative quantities are generally admitted by everyone and are generally received as correct, whatever interpretation is to be attached to these quantities'".

11. It may be of interest to note the history of the solution of linear indeterminate equation in Europe, as given by Dickson (History of theory of Nurhbers, Vol. II, New York, 1952, p.v.)

'An account of the method of solving ax+by=c (was) given by the Hindu Brahmagupta in the Seventh century. It was based on the mutual division of a and b as in Euclid's process of finding their greatest common divisor. Essentially the same method was rediscovered in Europe by Bachet de Meziriac in 1612, and expressed in the convenient notation of the development of a/b into a continued fraction by

Sauhderson in England in 1740 and by Lagrange in France in 1767. The simplest proof that the equation is solvable when a and b are relatively prime is that given byEuler in 1760'.

12. T.A. Saraswati Amma Ref (ld) p. 3.

13. To cite contemporary authorities, we have for instance Bertrand Russel recounting. "I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere"; or David Hilbert declaring "The goal of my theory is to establish once and for all the certitude of mathematical methods" (Both quotations cited in Reuben Hetsh, Some Proposals for Reviving the Philosophy of Mathematics, Adv/Math. 31,1979,31-50).

14. A recent book recounts how the continued Western quest for securing absolute certainty for mathematical knowledge originates from the classical Greek civilization, as follows:

The crisis (in the foundations of mathematics which became an important issue in the 20th century) was a manifestation of a long-standing discrepancy between the traditional ideal of mathematics, which we call the Euclid myth and the reality of mathematics, the actual practice of mathematical activity at any particular time... What is the Euclid myth? It is the belief that the books of Euclid contain truths about the universe which are clear and in dubitable. Starting from self-evident truths, and proceeding by rigorous proof, Euclid arrives at knowledge which is certain, objective and eternal. Even now, it seems that most educated people believe in the Euclid myth. Up to the middle or late nineteenth century, the myth was unchallenged. Every one believed it. It has been the major supporter of metaphysical philosophy...

The roots of the philosophy of mathematics as of mathematics itself are in classical Greece. For the Greeks, mathematics meant geometry, and the philosophy of mathematics in Plato and Aristotle is the philosophy of geometry. For Plato, the mission of philosophy was to discover true knowledge behind the veil of opinion and appearance, the change and illusion of the temporal world. In this task, mathematics, had a central place, for mathematical knowledge was the outstanding example of knowledge independent of sense experience, knowledge of eternal and necessary truths' (Philips J. Davis and Reuben Hersh, The Mathematical Experience, Boston, 1981, pp. 323-325)

15. To cite a historical instance, we only need to recall how the adherence to the Greek philosophy of mathematics had left Western mathematics in a very poor shape as for as arithmetic and algebra are concerned, for over a thousand and five hundred years. As a Western scholar remarked recently, We may certainly say that the Greeks have given the Western world geometry, but algebra and the number system has had to come from the Indian and Arabic world. There is every reason to uphold that this rejection of algebra by the Greeks was consciously done as not "fitting" the philosophy they subjected mathematics to' (W. Kuyk, Complementarity in Mathematics, Reidel 1977, p.71)

16. One philosopher of science has argued that this barrenness of the contemporary philosophy of mathematics can be traced to the basic epistemological position on the nature of mathematical knowledge espoused in the Western tradition:

'Under the present dominance of formalism the school of mathematical philosophy which tends to identify mathematics with its formal axiomatic abstraction and the philosophy of mathematics with metamathematics,. one is tempted to para phrase Kant: The history of mathematics, lacking the guidance of philosophy, has become blind, while the philosophy of mathematics, turning its back on the most intriguing phenomena in the history of mathematics has become empty... The history of mathematics and the logic of mathematical discovery, i.e. the phylogenies and the ontogenesis of mathematical thought, cannot be developed without the criticism and ultimate rejection of formalism...

But formalist philosophy of£mathematics has very deep roots. It is the latest link in the long (chain of dogmatist philosophies of mathematics. For more than two thousand years there has been an argument between dogmatists and sceptics. The dogmatists hold that - by the power of our human intellect and/or senses - we can attain truth and know that we have attained it. The sceptics on the other hand either hold that we cannot attain the truth at all (unless with the help of mystical experience), or that we cannot know if we can attain it or that we have attained it. In this great debate, in which arguments are time and again brought up-to-date, mathematics has been the proud fortress of dogmatism. Whenever the mathematical dogmatism of the day got a 'crisis', a new version once again provided genuine rigour and ultimate foundations, thereby restoring the image of authoritative, infallible, irrefutable mathematics, 'the only Science that it has pleased God hitherto to bestow on mankind' (Hobbes (1651], p.15). Most sceptics resigned themselves to the impregnability of this stronghold of dogmatist epistemology. A challenge is now overdue'. (I. Lakatos, Proofs and Refutations, Cambridge 1976, pp. 1-5)

17. Philip J. Davis and Reuben Hersh: The Mathematical Experience, (Boston, 1981, pp. 354-5).

18. The nature of the mathematical discourse fostered by the 'Euclidean methodology' is aptly described as follows:

'Euclidean methodology has developed a certain obligatory style of presentation. I shall refer to this as 'deductive style'. This style starts with a painstakingly stated list of axioms, lemmas and/or definitions. The axioms and definitions frequently look artificial [and mystifyingly complicated. One is never told how these complications arose. Tine list of axioms and definitions is followed by the carefully worded theorems. These are loaded with heavy-going conditions; it seems impossible that anyone should ever have guessed them. The theorem is followed by the proof.

The student of mathematics is obliged, according to the Euclidean ritual, to attend this conjuring act without asking questions either about the background or about how this sleight-of-hand is performed. If the student by chance discovers that some of the unseemly definitions arej proof- generated, if he simply wonders how these definitions, lemmas and the theorem can possibly precede the proof, the conjuror will ostracize him for this display of mathematical immaturity.

In deductive style, all propositions are true and all inferences valid. Mathematics is presented! as an ever-increasing set of eternal, immutable truths. Counter example, refutations, criticism cannot possibly enter. An authoritarian air is secured for the subject by beginning with disguised monster-barring and proof-generated definitions and with the full-fledged theorem, and by suppressing the primitive conjecture, the refutations, and the criticism of the proof. Deductive style hides the struggle, hides the adventure. The whole story vanishes, the successive tentative formulations of the theorem in the course of the proof- procedure are doomed to oblivion while the end result is exalted to sacred infallibility" (I. Lakatos, Proofs and Refutations, Cambridge, 1976)

19. In fact it was argued twenty-thirty years ago in the West, that the reform should go the other way - replace whatever informal mathematics that may be there in school curriculum etc., by axiomatic, set theory etc. The West might have by and large given up this project, but its followers elsewhere are still keen on cleansing the school curricula of irrelevancies.

20. See for instance, the following:

(a) Navjyoti Singh, A Comparative Study of the Foundation of Mathematics in Greece, India, China and the Modern West, PPST Bulletin No. 8,1985,53-73

(b) Chhatrapati Singh, The Philosophical Foundations of a General Theory of Numbers, Paper presented at the NISTADS Conference, Delhi 1984.

21. See for instance the following:

(a) D.H.H. Ingalls, Materials for the study of Navya- nyaya Logic, Harvard, 1951.

(b) D.C. Guha, Navya-nyaya System of Logic, Varanasi 1979.

(c) J.L. Shaw, Number: From Nyaya to Frege-Russel, Studia Logica, 41, 1982, 283-291.

(d) Roy W. Perret, A note on the Navya-nyaya Account of Number, Jour. Ind. Phil. 15, 1985, 227-234.

(e) B.K. Matilal, On the Theory of Number and Paryapti in Navyanyaya, JASB, 28, 1985, 13-21.

22. Ironically hundred years ago Frege himself laid the blame for the totally unsatisfactory character of arithmetic (that prevailed in his times and earlier) to the fact that its method and concepts originated in India:

'After deserting for a time the old Euclidean standards of rigour, mathematics, is now returning to them, and even making efforts to go beyond them. In arithmetic, if only because many of its methods and concepts originated in India, it has been the tradition to reason less strictly than in geometry which was in the main developed by the Greeks' (G.Frege, Foundations of Arithmetic, Tr. by J.L. Austin, Oxford, 1956, p.le)

23. This is only part of the story. The full picture is more complicated as described below (B.K. Matilal Ret 21 [el pp. 18,19):

'Ail numbers are recognized as objective realities, in fact objective properties resident in the things numbered. All numbers except unity or one-ness (according to some) are transitory entities created in such numbered objects and then destroyed when one of their crucial causal factors is destroyed. This crucial causal factor is however a cognitive, event - which is technically called apeksha budhi a conjunctive - count-oriented cognition. This cognitive event tacitly arises in the observer (counter) and may be expressed verbally as "this is one and that is one (which makes two)". This emergent "counting cognition" continues to exist until the observer or counter has perceived the two-ness, three-ness etc., in two things or three things. After such perceptions, this count-oriented cognitive event perishes (as all events must), and when such a crucial factor disappears there is no reason, as the Nyaya argument goes, for us to think that twoTness or three-ness exists. This two-ness is thus unique (particular) in two ways, due to its own unique causal history as well as due to the uniqueness of the items counted. Universals are however said to inhere in particulars and hence a universals of two-ness is also posited as resident by inherence in each instance of duality (as we count two objects).

The usual English number-words, one, two etc. have slightly ambiguous syntactical function. They are used mostly as adjectives or qualifiers; two mangoes, one man. We can also use "two is a number" etc. This substantial use of "two" can be designative of properties or locates (dharma). Hence to dispel such ambiguity we may follow the Sanskrit style and use "two-ness" or "duality" as designative of the property that we call number two.

One cannot fail to notice the strain of "subjectivity" in the above Nyaya conception of numbers as "Objective" properties. They are caused by a cognitive event that arises in the observer and the same event also accounts for its perception by the observer. They arise as particular occurant and disappear from the objective world as soon as the said cognitive event is over. They seem to be very strange sort of objective properties, if we can call them objective at all. They are neither mental and it may be wrong to call them material in the usual sense. Two points are suggested to explain this oddity. Firstly, here as elsewhere, Navya-nyaya does not choose to talk in terms of such a mental-material dichotomy. But the exponent of Navya-nyaya would resist all attempts to describe numbers as mental entities, if such descriptions mean that they are not "out there" in the things. Secondly, even a cognitive event in Nyaya is treated as an object and its causation, may deliver "objective" realities. In fact even the self in Nyaya can hardly be called "subject- dependent" or "subjective".

24. For an account of the problems associated with the notion of a "set", see Chatrapati Singh, What is a Set, Indian Law Institute Preprint 1985.

25. As one scholar has explained:

The translation of Navya-nyaya talks of properties into talk of classes seems gratuitous. Navya-nyaya confines itself to talk of universals and particulars, never introducing talk of I classes and members. In this sense Navya-nyaya is an intentional system. The term "intentional" can be a source or confusion, for it is variously used in a number of senses. Here I want to commit myself to no more than the claim that Navya-nyaya accepts properties as elements of the system, though it does not explicitly admit classes as such elements... Recent work in intentional logic, however, casts doubt upon the sanguine assumption that intentional arguments can always be rendered into extensional system... It is extremely important to recognize that Quine's claim that properties are reducible to classes is a philosophical thesis. It is, furthermore a highly controversial thesis... It leaves us with unattractive consequences such as the supposed identity of apparently distinct properties like "having a kidney" and "having a heart", since the instances of each are identical. There seem good reasons, then for not interpreting the Navya-nyaya account of number extensionally in terms of classes and numbers. (Roy. W. Perret, Ref. 21d, pp. 228-9).

26. G.T. Kneebone, Mathematical Logic and the Foundations of Mathematics, London, 1963, p. 117.

27. See for instance, M.D. Srinivas, The Indian Approach to Formal Logic and the Methodology of Theory Construction: A Preliminary view, PPST Bulletin No. 10, 1986, 32-59.

28. G.T. Kneebone, Ref. 26, p. 326.

29. 29. G.T. Kneebone, Ref. 26, p. 326.

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