Book Review:

Bhaskaracharya's Lilavathi

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May 16, 2005

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Part I

A Brief History of Indian Mathematics

Chapter 1

Introduction

In this work, I review Bhaskaracharya's classic work **Lilavati**. I begin by a brief history of Indian Mathematics. It is by no means complete, but serves the purpose of establishing the context in which Bhaskaracharya's monumental work can be reviewed.

1.1 A Brief History of Classical Indian Mathematics (up to Bhaskara II):

Laplace (1749-1827), the French mathematician once remarked - *"The ingenious method of expressing every possible number using a set of ten symbols (each symbol having a place value and an absolute value) emerged in India. The idea seems so simple nowadays that its significance and profound importance is no longer appreciated. Its simplicity lies in the way it facilitated calculation and placed arithmetic foremost amongst useful inventions. the importance of this invention is more readily appreciated when one considers that it was beyond the two greatest men of antiquity, Archimedes and Apollonius"*.

The origin of Indian Mathematics can be traced back to the descriptions of the geometry for altar constructions as found in the Vedic mythology text – the *Shatapatha Brahmana* and the *Taittiriya Samhita*. Mathematical astronomy of India dates back to 3rd century B.C. Mathematics and especially Geometry was needed to support developments in astronomy in India.

Indus valley civilization (Harappan) in 2500 B.C. already had mathematical descriptions of weights and measures. The Harappans had systematic weights and measures, which were fractional in nature − 0.05, 0.1, 0.2, 0.5, 1, 2, 5, 10, 20, 50, 100, 200, and 500. One Indus inch had a unit of 1.32 inches (of the present day).

The Vedas which are believed to have been composed in 1500 B.C. and 800 B.C. (in sanskrit) contain the *Sulbasutras*. These are appendices to Vedas giving rules for construction of altars. These were composed by *Baudhayana* in 800 B.C. Other notable contributions to geometry of altars were made by *Manava* in 750 B.C. and *Apastamba* in 600 B.C. and *Katyayana* in 200 B.C. These men were mainly priests and scholars and not mathematicians in the modern sense. *Panini* lived in the same period who was a grammarian par excellence. What has grammar got to do with Mathematics? It is now clear that this has connections to mathematical formal language and computer science.

Brahmi Numerals which subsequently got modified to the present day numerals were invented in Indian in the 1st century A.D. The *Jains* started replacing Vedic religion in 6th century B.C. From this period to 500 A.D. (*Aryabhata I*) is considered dark period of Indian Mathematics. It was around 150 B.C. that *Jaina* mathematics flourished. They were quite advanced to their counterparts (the Vedic) because they even considered orders of infinity and also had the concept which was akin to the *logarithm* to base-2. The famous *Bhakshali Manuscript* consists of these *Jaina* writings, but it's date is controversial.

500 A.D. was in some sense the beginning of the era of Indian Mathematics with outstanding contributions by *Aryabhata*. He replaced the *Rahu* and *Ketu* theory with a more modern theory of eclipses. He headed the research center in mathematics and astronomy at Kusumapura (NE). Ujjain was another center which was then the home

of *Varahamihira* who was also a famous mathematician. Both schools were involved in development of numerals and of place-valued number systems.

Brahmagupta was the next significant figure in the 7th century A.D. at Ujjain who made remarkable contributions to negative numbers and zero, integer solutions to indeterminate equations and interpolation formulas (for computation of sine tables). He was the earliest astronomer to have employed the theory of quadratic equations and the method of successive approximations to solving problems in spherical astronomy.

Mathematics in India at that time was family-based. Mathematical education was largely restricted within the family and there was not much scope of innovation. Father passed on the commentaries to his son, who pursued it. The role of women in mathematics during those times was highly restricted or even probably non-existent. Religion also played a key role. Mathematical beliefs were tantamount to religious beliefs and changing religious beliefs was not acceptable. The role of commentaries was important because mathematicians wrote commentaries on their own work and the work of their predecessors. These commentaries got transferred in disclipinic succession.

One of the characteristics of Indian system of mathematics and science was the lack of sufficient observations. *Parameshwara* of the 14th century A.D. was the first mathematician/astronmer who made systematic observations over many years. Indian system of mathematics was largely rule-based or algorithmic in nature. There were no use of notations or symbols and these were in terse verse form. Even numerals were not used in these verses. Further, mathematics was largely a computational tool for developing astronomy. It was meant as a tool to enable greater simplicity and clarity in understanding astronomical facts and phenomena.

The next significant figure was *Bhaskara I* (7th century A.D.). He was a contemporary of *Brahmagupta* at theUjjain center and led the *Asmaka* school. This school would have the study of the works of Aryabhata as their main concern and certainly Bhaskara was a commentator on the mathematics of Aryabhata. More than 100 years after Bhaskara, the astronomer *Lalla*, lived another commentator on Aryabhata.

Bhaskara II (1114 - 1185 A.D.) also known as *Bhaskharacharya* was born in Maharashtra. He is considered as an outstanding poet-mathematician of his times. He was the head of the astronomical observatory at Ujjain, where other famous Indian mathematicians including Brahmagupta had studied and worked previously. He worked on algebra, number systems, and astronomy. He wrote beautiful texts illustrated with mathematical problems and he provided the best summary of the mathematics and astronomy of the classical period. He made fundamental contributions to the development of number theory, the theory of indeterminates infinite series expressions for sine, cosine and tangent, computational mathematics, etc. 200 years after Bhaskara did any significant work happened in Indian Mathematics.

Bhaskara was a great poet and had mastered eight volumes on Grammar, six on medicine, six on logic, five on mathematics, four vedas, a triad of three ratnas, and two *Mimamsas*. Bhaskara produced six works during his lifetime: **Lilavati, Bijaganita, Siddhantasiromani, Vasanabhasya of Mitaksara, Brahmatulya, and Vivarana**. These were all books on math or astronomy, with some of them being commentaries on his own works or that of others. Bhaskara was excellent at arithmetic, including a good understanding of negative and zero numbers. He was also good at solving equations and had an understanding of mathematical systems, years ahead of his European peers.

The other notable mathematician post-Bhaskara was *Madhava* from Kerala (1350-1425 A.D.). He invented Taylor series and gave an approximation of π to 11 decimal places. It is remarkable that he also gave the error or reminder term for the Taylor's expansion of $\frac{\pi}{4}$ which was rediscovered by Newton in 1676. It is said that the 2 most innovative Indian mathematicians of the classical period were Bhaskaracharya and Madhava.

1.2 About the translation

For the book review, we use **'Lilavati of Bhaskaracharya (A Treatise of Mathematics of Vedic Tradition)'** translated by Krishnaji Shankara Patwardhan, Somashekhara Amrita Naimpally and Shyam Lal Singh [1].

The above authors have translated the Marathi work of Professor Phadke tiled *'Lilavati Punardarshana' – A New Light on Lilavati* written in 1971. He took great pains in writing the Marathi translation with comments and explanation. The authors acknowledge Professor Phadke for his excellent translation and his attempt to explain the rationale of Lilavati in terms of modern mathematics.

1.3 Lilavati

Lilavati is the first part of Bhaskaracharya's work *Siddhantashiromani* which he wrote at the age of 36. Siddhantashiromani consists of four parts namely – 1) Lilavati 2) Algebra 3) Planetary motions and 4) Astronomy.

Lilavati has an interesting story associated with how it got its name. Bhaskaracharya created a horoscope for his daughter Lilavati, stating exactly when she needed to get married. He placed a cup with a small hole in it in a tub of water, and the time at which the cup sank was the optimum time Lilavati was to get married. Unfortunately, a pearl fell into the cup, blocking the hole and keeping it from sinking. Lilavati was then doomed never to wed, and her father Bhaskara wrote her a manual on mathematics in order to console her, and named it Lilavati. This appears to be a myth associated with this classical work. Lilavati was used as a textbook in India in Sanskrit schools for many centuries. Even now, it is used in some Sanskrit schools.

1.3.1 Contents of Lilavati

Lilavati mainly deals with what we call as 'Arithmetic' in today's mathematical parlance. It consists of 279 verses written in Sanskrit in poetic form (terse verses). There are certain verses which deal with Mensuration (measurement of various geometrical objects), Volume of pyramid, cylinders, heaps of grains etc., wood cutting, shadows, trigonometric relations and also on certain elements of Algebra such as finding an unknown quantity subject to certain constraints using the method of supposition.

Bhaskaracharya wrote this work by selecting good parts from Sridharacharya's *Trishatika* and Mahaviracharya's *Ganitasarasamgraha* and adding material of his own. Lilavati became quite popular in India during the time it was first composed. Handwritten copies of Lilavati replaced most other prevalent texts of those times and eventually reached other countries.

1.3.2 About this review of Lilavati

The next part of this review examines some of the key features of Lilavati as translated in the book. I have divided the review itself into 4 chapters namely:

1. Arithmetic

2. Algebra

3. Trigonometry and Geometry

4. Discrete Mathematics

This is a logical division as per modern mathematics and not to be mistaken as the division made by Bhaskara. In the actual work, the Algebra section is quite mixed with

Arithmetic. In fact, Lilavati does not contain much of algebra, at least not explicitly. Also, the verses which deal with Permutations is separate from that of Combinations. The former appears before the verses on Geometry and the later appears after. Hence, this division which I have made is an artificial one and is for convenience only.

Part II

Lilavati: A review

Chapter 2

Arithmetic

This chapter reviews the verses of Lilavati on Arithmetic. The first verse of Lilavati is an invocatory verse on Lord Ganesha, as it was customary in those days before the beginning of any auspicious event. Here the word *Pati* is used for Arithmetic. It literally means *slate* mathematics. The verse also claims that this work is loved by discriminating people because of its *clarity*, *brevity* as well as *literary flavour*. This shows the literary significance of the work.

2.1 Definitions

Following the invocation, the next 9 verses deal with definitions of measurement units for various *things*. Firstly, he defines the various units of money which were in vogue during those days. This is followed by measures of gold, units of length, measures of grain in volume and lastly the measure of time. This indicates that the text is quite *formal* in treatment. Also, the fact that Arithmetic is directly related to commerce is evident. It sets the tone of the work not as an *abstract* piece but rather one of practical significance in day-to-day applications. This contrasts with other works of the time (western ?) where mathematical texts were not necessarily justifying their use in everyday life. There is a reference to the measure of grains as being *Turkish* in verse 9. The author infers that his verse must be an *inserted* verse since there was no influence of Muslims either in the north or in the south during Bhaskaracharya's times and that it was unlikely that they were in common use.

2.2 Definition of positional notation

The next two verses define the all-important positional notation of digits and their values. It clearly states that the value of digits increase by a factor ten from right to left. The highest value defined is $parardha = 10^{17}$. The fact that Bhaskaracharya was an astronomer probably justifies the need for such high magnitudes which may be required for describing large celestial distances. The author claims that the highest value defined in Sanskrit (not in this work) is 10^{140}. It is not clear as to what would be the *use* of such a high number because the number of atoms in the universe is only about 10^{80}. It is also interesting to note that absolutely no reference to numerals of any kind is explicitly mentioned, though it is known that Brahmi numerals which were invented in the first century A.D. were used.

2.3 Addition and Subtraction

The next few verses describe the *method* (or technique or algorithm) to add two numbers in the positional system. This is the characteristic of the entire work - the verses describe the method in an algorithmic fashion. The verse itself is *terse* (brevity is the soul of the wit ?) and there is no elaborate explanations of any kind. This is very contrasting to the scientific and mathematical works of the Greeks which are usually verbose.

An important thing to observe is that the verse claims that both addition and subtraction could be performed place-wise either from right to left or vice-versa. There also seems to be no mention of a *carry-over* whenever the sum of digits exceeded 9 (it was a base-10 system). However, it is possible that these were left as an excersize to the *student* to work-out the details. It is also possible that there was an intermediate step instead of a carry-over. An example is demonstrated here:

	tens	units
	2	6
+	8	7
=	10,	13

As it can be seen, in the above addition (either from left-to-right or right-to-left), carry-over is not employed. The sum for both the units and tens positions have exceed 9, but are retained as it is. There is a comma placed in order to identify the positional notation. This makes intuitive sense because 10 in the tens place is not allowed, but all it means is $10 \times 10 = 100$. Similarly, 13 in the units place is not allowed, but it means just 13. Hence, we perform the addition of these two as the next step:

	hundreds	tens	units
	1	0	0
+		1	3
=	1	1	3

This procedure is repeated until the final result has each digit < 10 in the positional notation. In the above example, we stop after the second step.

Although this seems cumbersome and requires more steps, there are two advantages:

1. This procedure could be performed either from left-to-right or from right-to-left and would ultimately yield the same result. This is not true with the modern carry-over method. In fact, in this method, addition can be arbitrarily performed starting from any positional value and in any order. The result would be the same. The carry-over

method requires the operation to be performed strictly from right to left.

2. For subtraction, a slight modification is necessary to incorporate this method. Assume that we wish to subtract 36 from 10000. We will have to split 10000 as $9999 + 1$. We could perform $9999 - 36$ in the same way as above to yield 9963. We then perform $9963+1 = 9964$ to arrive at the final answer. It is quite possible that Bhaskaracharya performed such splitting because elsewhere in giving methods for *multiplication*, he performs such splitting of numbers to simplify the procedure.

This is purely a speculation on my part. The illustrious Bhaskaracharya and his students may have been aware of the carryover method.

2.4 Multiplication and Division

Bhaskaracharya gives 5 different methods for multiplication. They involve various tricks like splitting the multiplier into two *convenient* parts and multiplying the multiplicand by each of the two parts and adding the results; factoring the multiplier, multiplying by each factor and then summing up the results etc. Another interesting characteristic of each of these methods is that Bhaskaracharya challenges the student following the introduction of each method by giving a problem/s to test the understanding of the student. Solutions are not provided but these examples are straightforward. He typically expects all the methods to be used to ensure that the solution is consistent. The verses are also quite poetic and beautiful. The use of poetic language typically involves the use of such adjectives and similies as – ‘ O! Auspicious girl with lovable eyes of a young dear’, ‘Oh Friend!’, ‘My beloved’, ‘Deer-eyed’, ‘Fickle-eyed’, ‘Oh! you intelligent gilr Lilavati’ etc. This clever use of language is partly teasing but also engaging and challenging the student intellectually. This is quite contrasting to modern mathematical text books which are always in *prose* form and quite dry. Elsewhere, Bhaskaracharya employs *humour* quite effectively which is also lacking in present day *text-books*. It should be remembered that *Lilavati* served as a text-book for nearly 800 years in different parts of India. This is something worth emulating in order to make education of mathematics *interesting*, *exciting* and also *fun*.

Bhaskaracharya gives only one method for division which is the same as the modern method of finding the product of the divisor with the largest integer such that it can be subtracted from the extreme left hand digits of the dividend. This integer is the first digit of the quotient. He also talks about removing any common *factors* of the dividend and divisor. This shows that he probably knew the *Fundamental Theorem of Arithemtic* that every integer can be factored *uniquely* into the product of *prime* factors. He does not however mention *primes* explicitly, but he does say that '*if it is possible to factorize the number*' and so on indicating that he probably had some idea about primes. Moreover, Euclid's work on *primes* including the famous proof of *infinitude of primes* was well documented in his magnum opus *'Elements'* which was written in 300 B.C.

2.5 Square root and n^{th} root

The *algorithm* for square root is nearly the same which is taught in modern mathematics. The only difference is that Bhaskaracharya groups the digits only *one* at a time instead of doing two at a time as modern mathematics does. The advantage of Bhaskaracharya's method is that the same procedure is extendible to n^{th} root. This shows that the methods which Bhaskarcharya and his predecessors like Brahmagupta and Aryabhata developed were not just aimed at solving the *specific* instance but they were also interested in methods which could be *generalized*. Sometimes speedness was traded for generality as will be evident in later examples also.

These methods are rarely used these days, thanks to the invention of *Logarithms* which makes these operations far easier. Logarithms were invented only later.

2.6 Fractions

It seems that Bhaskaracharya did not know about *decimals*. However, he knew about *fractions* and made an extensive study of them. He gives eight operations on fractions. He also knew about Least Common Multiple (LCM) and Highest Common Factor (HCF) which are needed for various operations on fractions. He has a humourous example of a miser giving a beggar, $(\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{1}{5} \times \frac{1}{16} \times \frac{1}{4})^{th}$ part of a *dramma* (16 *drammas* make one *niska* which is one silver coin) which amounts to $\frac{6}{7680} = \frac{1}{1280}$ which is equivalent to one *Kavadi* or *Cowrie*, the lowest denomination possible thus justifying the miser's virtue (or vice ?).

Bhaskara gives methods for multiplication of fractions, addition of fractions, divisions, square-root, cube-root, squares, cubes etc. These methods are more or less extensions of the earlier ones on integers, only suitably modified.

2.7 On Infinity and Eight rules of Zero

Bhaskaracharya seemed to have known the importance of zero, not just in positional notation, but also as a number. He has special verses describing the *peculiar* properties of zero. He lists eight rules such as $a + 0 = 0$, $0^2 = 0$, $\sqrt{0} = 0$, $a \times 0 = 0$ etc. The interesting aspect of this verse is the *definition* of infinity or *Khahara* as a *fraction* who's denominator is zero. In other words, $\frac{a}{0} = \infty$. He knew that whenever a number was divided by zero, it lead to a problem. He gives a verse on the nature of this infinity as follows (copied from translation):

> "There is no change in infinite (khahara) figure if some thing is added to or subtracted from the same. It is like there is no change in infinite Visnu (Almighty) due to dissolution or creation of abounding living beings".

This definition of infinity is definitely wrong but the properties are right (as against what modern mathematics defines). During his time, there were other astronomers and mathematicians who did not believe in the immutability of infinity. For example, *Jnanaraja* (1503 A.D.) says that infinity does not remain immutable when something is added to it or subtracted from it. However they used *Khahara* for infinity. It seems that they did not have any other symbol or notation for infinity. They used the $\frac{a}{0}$ as a notation for infinity quite safely for further calculations or reversing the process of mathematical operations as and when needed.

2.8 Reverse process and the concept of limits

Bhaskaracharya elaborates *inverse* operations for back-calculations for obtaining an unknown quantity from known ones in a mathematical expression. As an example, in order to find a certain number which when added by 10, 4 subtracted from $\frac{2}{5}^{th}$ of the sum so obtained to yield a result of 12, the inverse process is employed. 4 is first added with 12 to yield 16 and then 16 is divided by $\frac{2}{5}$ to yield 40. Finally, 10 is added to 40, to yield the original number 30 as the correct answer. This reverse method can be applied to all mathematical operations.

Bhaskaracharya specifically mentions that if a certain number is multiplied by zero and also divided by zero, it should not be treated as either zero or infinity. It should be retained as is for further calculations. He demonstrated this with an example:

> " A certain number is multiplied by 0 and added to half of the result. If the sum so obtained is first multiplied by 3 and then divided by 0, the result is 63. Find the original number. It is obtained by 'reverse' (*Viloma*) process."

For the above example, the expected solution is as follows: $\frac{(x\times 0+\frac{1}{2}\times 0)\times 3}{0} = 63$ implies $\frac{9x}{2} \times \frac{0}{0} = 63$ which implies $x = 14$.

The author claims that this closely borders the definition of a limit, because strictly speaking the last step of the above process is similar to saying $\lim_{h \to 0} \frac{(x \times h + \frac{1}{2} \times h) \times 3}{h} = 63$ which implies $x = 14$.

I do not share the same degree of excitement as the author when he says that this is the concept of the limit in traditional language and hinting that if the traditional Indian savants had understood it properly they could have invented differential calculus. I do agree that this was definitely a step forward in the right direction. However, the concept of limits and the theory of Calculus needed much more insight and machinery and had to wait for the genius of Newton and Leibniz.

2.9 Ratio and Proportions

Bhaskara extensively treats direct and inverse proportions. He cites several problems where the *rule of three* is applicable. This states that if $a : b :: c : d$ where a and c are of the same kind and b and d are of the same kind. d, the unknown can be found by the formula $d = \frac{b \times c}{a}$. This holds only for *direct proportion* and Bhaskara explicitly mentions this and the inverse proportion relation also. He extends this rule for the rule of five, seven, nine etc.

2.10 Simple Interest, Progressions

Bhaskara deals with simple interest with apparent ease. He does not say anything about compound interest. He also gives a thorough treatment of arithmetic progressions (A.P) and geometric progression but not harmonic progression. He gives the direct and inverse formulae for finding the sum of series, the last term, the constant difference for an A.P.

Chapter 3

Algebra

In this chapter, we review methods in Lilavati pertaining to Algebra. Bhaskaracharya does not explicitly use any language for mathematics. This may be the biggest handicap of the Indian system for it's development. As a result of this, algebra suffered greatly. The sheer manipulation of symbols can produce new results as we witness in modern mathematics and algebra. This was solely lacking in the Indian system because of the lack of a mathematical language. It is quite commendable that Bhaskaracharya and other Indian mathematicians developed all of their mathematics by intuition and empirical understanding rather than by random manipulations of symbols. Bhaskara and others of the classical Indian tradition never provided *proofs*. However, it is quite clear that they did have a certain *rationale* in their minds when they developed these results. They probably did not consider it important to show the proof because the method was quite obvious and self-evident. This is similar to the way mathematics was pursued by Gauss and his contemporaries. Gauss believed that intuitive ideas which helped a proof were like scaffolding of a building under construction and these necessarily had to be removed after the building was completed.

3.1 Squares and Cubes

Bhaskaracharya knew the elementary algebraic identities such as $(a+b)^2 = a^2 + b^2 + 2ab$, formula for a cube, $a^2 - b^2 = (a+b)(a-b)$ and many other complicated relationships. Pascal's triangle was also known. It was called as *Khandameru*.

3.2 Quadratic Equations

Bhaskaracharya deals with the quadratic equation $ax^2 + bx + c = 0$. Specifically he chose $a = 1$ and $c < 0$. This is because the discriminant $b^2 - 4ac$ turns out to be positive and the roots of the equation are *real*. Bhaskaracharya was not familiar with imaginary numbers (first known as *impossible* numbers). However, for the positive discriminant case, Bhaskaracharya gives the correct expression for the roots. In this section, Bhaskaracharya gives a very interesting puzzle from the epic Mahabharata where Arjuna uses a certain number of arrows (say x) to destroy the horses of Karna, a certain number to destroy his chariot, flag, bow and to cut off his head. The solution of the puzzle is the root of a quadratic equation. It is interesting to note that, Bhaskaracharya always solves for x^2 first and not x directly. This is because there was no symbol for *surds* (square root) and so he takes the square root of x^2 to obtain the value for x. He always takes the positive value of square root. It is not clear whether he knew that a square root had negative values.

There are no other methods on algebra though there is a use of some algebra in some of the methods he describes on trigonometry and geometry which is discussed in the next chapter.

Chapter 4

Trigonometry and Geometry

In this chapter, we review the verses in the Lilavati pertaining to Trigonometry and Geometry.

4.1 Triangles

Bhaskara starts with the definition of the sides of a right angled triangle. He then states the Pythagoras theorem (without proof). The author claims that it was known in India since the time of *Sulvasutrakaras* (3000-800 B.C.) whereas Pythagoras published it in 560 B.C.

Bhaskaracharya gives a method for finding an *approximate* square root of a number which is not a perfect square. To compute (approximately) $\sqrt{\frac{a}{b}}$, choose a large square number x. Then compute approximately $\sqrt{abx}$ and divide by $b\sqrt{x}$.

Bhaskara then deals with various problems related to different situations of right-angled triangles such as finding the two sides when the hypotenuse is given.

Bhaskara states that it is impossible for one side of a triangle to be greater than sum of the other two sides. This *Triangle Inequality* is valid only in Euclidean geometry.

4.2 Mensuration of Polygons and other objects

Bhaskara gives formula for finding the area of a quadrilateral (both cyclic and acyclic). He also gives the formula for finding the diagonal of a quadrilateral. He also deals with trapezium, disk, sphere etc. He deals with chords of a circle extensively.

4.3 Volume

Bhaskara gives methods for determining volume of a pyramid and its frustrum and the volume of a prism. He also gives practical applications of finding the cost for cutting wood in a particular shape (frustrum of a cone) and its area calculation. He also gives methods to calculate volume of a heap of grain.

An important observation to make is that there are no diagrams or figures for illustrating any of these geometrical methods. This is most peculiar because geometry is treated in the same way as arithmetic and algebra. One can infer that the lack of diagrams was another handicap for the development of Indian mathematics, especially geometry.

He also deals with lengths of shadows.

Chapter 5

Discrete Mathematics

5.1 Permutations, Combinations and Partitions

Bhaskara gives a fair amount of treatment of what we call today as *Discrete Mathematics*. He correctly gives the formulae for $n!$, ${}^nP_r$, ${}^nC_r$ etc. He specifically mentions that the study of combinatorial analysis is useful in prosody to discover all possible meters (he also gives a puzzle on similar lines), in architecture, medical sciences, Khandameru (Pascal's triangle), chemical composition etc. He further claims that he is omitting these applications for the sake of brevity. This shows that Bhaskara was very much interested in applications of Mathematics and was not a *pure* mathematician. In those days, it was probably uncommon to pursue mathematics for it's own sake. This also shows that Bhaskara knew the importance of mathematics in a wide variety of applications. Whether he applied it to other disciplines (other than astronomy) is not known.

Bhaskara illustrates the principle of finding the number of permutations by an interesting puzzle:

> " Lord Shiva holds ten different weapons, namely a trap, a goad, a snake, a drum, a potsherd, a club, a spear, a missile, an arrow and a bow in his hands. Find the number of different Shiva idols. Similarly, solve the problem for Vishnu

idols; Vishu has four objects: a mace, a disc, a lotus and a conch."

For the first example, there are $10! = 3628800$ possible Shiva idols and in the case of Lord Vishu, there are $4! = 24$ different idols. The author makes an interesting observation that in the daily ritual (*sandhyavandanam*), there are 24 different names of Lord Vishnu.

Bhaskara also deals with repeated digits and their combinations. He also gives an elementary problem in Partitions. This shows the diversity of topics covered by Bhaskara.

5.2 *Kuttaka* or Indeterminate analysis

This is towards the end of *Lilavati*. Indeterminate analysis is the problem of finding integer solutions to x and y in the equation $ax + by = c$ where a, b and c are all integers. The method to do this was called as *Kuttaka* which means "to beat the problem into powder". In other words, the solution which was developed by Brahmagupta and later by Bhaskaracharya involved successively simplifying the problem in an iterative process and then solving it. This reminds of powdering a larger object into smaller pieces first before powdering these pieces into finer pieces and so on. These are also known as *Diphontine Equations*.

Bhaskaracharya provides a method for finding the solution which makes use of the *Euclidean Algorithm* for finding the Greatest Common Divisor (G.C.D). The *Kuttaka* method is said to be an important contribution of Bhaskaracharya.

5.3 Discussion and Concluding remarks

Lilavati, Bhaskaracharya's monumental work is not only a literary treatise but also occupies a distinguished and honored place in the history of Mathematics. It is a testimony to the extra-ordinary mathematical acumen of Bhaskaracharya, who is regarded as one of the most innovative mathematician of India of his era. He was also an excellent teacher as indicated by the teasing and pleasing verses through which he tests his disciple's abilities to solve mathematical problems. Bhaskaracharya may not know anything about what a *Proof* is, which is the most important part of any modern mathematical work. However, his terse verses which contain the algorithmic rules and the generalizations which he recommends indicate that there may have been a certain *rationale* or *reasoning* in the minds of the mathematicians of those times. Moreover, his methods are not vague by any means and are quite precise. They probably relied more on *intuition* and did not feel the necessity of providing the *rationale* they had, if any. This is best exemplified by the genius Srinivasa Ramanujan of our times who's intuitive leaps of imagination were closer to Bhaskaracharya in spirit. Ramanujan's idea of a proof was very sketchy and when Hardy and Littlewood asked him why a particular result of his was true, Ramanujan would reply that it was true because he knew it (or rather it occurred to him in a flash). This was by no means a proof, but it shows the power of intuition, the ability to find new patterns without the shackles of formalism.

However, this was also the limitation of Indian Mathematics. It did not flourish to it's full potential. There are several reasons for this. There were no symbols or notation invented to handle mathematical objects and this was a huge handicap. This is a very important process which leads to the *unreasonable effectiveness of mathematics*. The language in which it was written - Sanskrit - was difficult and only the very well learned scholars cold decipher the poetic verses. The other reason for its loss of popularity was the lack of a *proof* or elaboration of the *rationale* behind the methods. Western mathematics and science very much demanded not just the result but *why* the result was true, if it was true at all. This may seem too much to ask for, but it was useful, because it guaranteed the determinism of Mathematics. All mathematical knowledge became sacrosanct, thanks to the rigorous demand of *proof*. It became sacred and deterministic to the extent that many people would take to it because they could not face the realism of the world and other sciences. Paul Erdos, the Hungarian mathematical genius of our times was an example of this kind. He took to mathematics because it was the only thing in this world that was guaranteed to be true, whatever truth meant to him. Nothing else appealed to him because of their innate uncertainty barring mathematics. Another huge handicap of Indian mathematics was the lack of illustrations or diagrams. As a result of this, Indian treatment of geometry was only very elementary. It is possible that they used some diagrams while solving problems, but this was never communicated in their works. What these ancient Indian mathematicians seem to excel was in their intuitive way of thinking and the algorithmic approach to mathematics. This is quite useful, especially in the computer age where such algorithmic approach would have received a huge support and would have been an advantage. Another advantage of Indian mathematics was the conciseness of the documentation. This is a very good example of *compression* of mathematical ideas. This was partly the reason that it was easily copied and transferred either by hand or orally. Remember, there was no printing available during those times.

The last verse of Lilavathi demonstrates the poetic brilliance of Bhaskara:

> " (Lass) Lilavati is born in a respectable family, stands out in any group of enlightened persons, has mastered idioms and proverbs. Whomsoever she embraces will be happy and prosperous."

However, the same verse could also be interpreted as:

> "This Lilavati clearly explains *fractions*, *simple fractions*, *multiplication* etc. It also beautifully describes problems in day-to-day transactions. Rules are transparent and examples are beautifully worded. Those who master this Lilavati will be happy and prosperous."

Bhaskaracharya was given the title of *'Ganakacakrachudamani'* which means *'A Crest Jewel among Mathematicians'*. Now we know why.

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*Thanks to Prof. Balachander Rao for giving useful advice.